Improved Understanding of Transonic Flutter:  
a Three Parameter Flutter Surface

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An understanding of transonic flutter is often critical for highspeed aircraft development. A presentation of the transonic flutter velocity as a function of the Mach number and mass ratio is shown here to provide many advantages. Such a presentation offers new insights when comparing computational and wind tunnel flutter results. The benefits of such a presentation are also evident in parameter studies. Finally, the subject of flutter similarity rules for airfoils of different thicknesses is also addressed.

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
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<tbody>
<tr>
<td>a</td>
<td>nondimensional location of airfoil elastic axis, ( a = c/b )</td>
</tr>
<tr>
<td>( a_0 )</td>
<td>stagnation speed of sound</td>
</tr>
<tr>
<td>b, c</td>
<td>semi-chord and chord, respectively</td>
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<tr>
<td>( e )</td>
<td>location of airfoil elastic axis, measured positive aft of airfoil mid-chord</td>
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<tr>
<td>( h )</td>
<td>airfoil plunge degree-of-freedom</td>
</tr>
<tr>
<td>( I_\alpha )</td>
<td>second moment of inertia about elastic axis</td>
</tr>
<tr>
<td>( L )</td>
<td>aerodynamic lift</td>
</tr>
<tr>
<td>( K_{h}, K_{\alpha} )</td>
<td>airfoil plunge stiffness and torsional stiffness about elastic axis, respectively</td>
</tr>
<tr>
<td>( M )</td>
<td>freestream Mach number</td>
</tr>
<tr>
<td>( M_{e,a} )</td>
<td>aerodynamic moment about elastic axis</td>
</tr>
<tr>
<td>( m )</td>
<td>airfoil sectional mass</td>
</tr>
<tr>
<td>( r_\alpha )</td>
<td>radius of gyration of airfoil about elastic axis, ( r_\alpha^2 ) is identical to ( I_\alpha/mb^2 )</td>
</tr>
<tr>
<td>( S_\alpha )</td>
<td>first moment of inertia about elastic axis</td>
</tr>
<tr>
<td>( T/T_0 )</td>
<td>absolute temperature ratio</td>
</tr>
<tr>
<td>( U )</td>
<td>freestream velocity</td>
</tr>
<tr>
<td>( V )</td>
<td>reduced velocity, ( V ) is identical to ( U/\omega_\alpha b )</td>
</tr>
<tr>
<td>( V_\mu )</td>
<td>flutter speed index, ( V_\mu ) is identical to ( U/\sqrt{\mu} \omega_\alpha b )</td>
</tr>
<tr>
<td>( x_\alpha )</td>
<td>airfoil static unbalance, ( x_\alpha ) is identical to ( S_\alpha/mb )</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>airfoil pitch degree-of-freedom</td>
</tr>
<tr>
<td>( \Delta X )</td>
<td>relative difference of two variables, ( X_1 ) and ( X_2 ), ( \Delta X ) is identical to ( (X_1 - X_2)/X_1 )</td>
</tr>
</tbody>
</table>

\( \delta \) = thickness of airfoil  
\( \varepsilon \) = frequency dependent exponent in \( \psi \)  
\( \mu \) = mass ratio, \( \mu \) is identical to \( m/\pi \rho b^2 \)  
\( \rho \) = freestream density  
\( \Sigma \) = compatibility surface (relates \( V \) and \( M \))  
\( \chi \) = transonic aerodynamic similarity parameter  
\( \psi \) = transonic flutter similarity parameter  
\( \omega_1, \omega_2 \) = frequency and reduced frequency based on airfoil chord, \( \omega \) is identical to \( \omega_c/U \)  
\( \omega_{x_1}, \omega_{x_2} \) = uncoupled natural frequencies of pitch and plunge DOF  
\( \omega_{1,2} \) = coupled structural natural frequencies

Introduction

Bendiksen\(^1\) has made several interesting and important observations about transonic flutter boundaries as determined in a typical transonic wind tunnel test. In particular he has noted that single degree of freedom flutter may occur at low values of air density or dynamic pressure when the test is conducted at a fixed Mach number by increasing the wind tunnel stagnation pressure and therefore air static density and dynamic pressure. The present authors have conducted a rather extensive parameter study\(^2\) in transonic range for a two degree of freedom airfoil in plunge and pitch and this work is extended here to further consider Bendiksen’s observations as well as other physical parameters not explored in Ref. [2].

Bendiksen’s results are reconfirmed and the need for careful identification of the flutter boundary in the transonic range and the need for extensive data sets to define such boundaries are emphasized. Fortunately newly emerging methods that provide much more efficient calculation of transonic flutter boundaries are now becoming available\(^2-4\). Also the results of Bendiksen and the present results suggest the need to reconsider how transonic wind tunnel flutter tests are conducted. For example, it may be desirable to
fix stagnation pressure and vary Mach number rather than vice versa to determine the experimental flutter boundary when testing in the transonic flutter regime.

**Governing Equations and Computational Method**

Consider a “typical” two degree-of-freedom (DOF) airfoil section with the equations of motion:

\[
\begin{align*}
    m \ddot{h} + S_\alpha \dot{\alpha} + K_h h &= -L \\
    S_\alpha \ddot{\alpha} + I_\alpha \dot{\alpha} + K_\alpha \alpha &= M \text{e.a.}
\end{align*}
\]

Here the left-hand side terms represent a linear structural model approximation for the plunge and pitch coordinates. The right hand side terms represent the aerodynamic loading terms, which for this study are based upon the harmonic balance approach applied to a discrete CFD model of the inviscid Euler equations. A summary of the method, its application to parametric flutter analysis and convergence study are given in a recent work of the authors\(^2\). For a more detailed description of the inviscid computational fluid dynamic harmonic balance aerodynamic Euler based method see Ref. [5].

**Flutter Boundary Surface**

Computational results were initially obtained for a NACA 65A004 airfoil and eighteen Mach numbers between 0.3 ≤ M ≤ 1.0 in the reduced frequency range 0.0 ≤ \(\tilde{\omega}\) ≤ 0.8. The three-dimensional parameter \((V_f, M, \mu)\) surface for the flutter is shown in Fig. 1a for a range of mass ratio from 20 ≤ \(\mu\) ≤ 200.

A possible flight test flutter trajectory is denoted by the line \(AB\). It was calculated in the following manner. Suppose the airfoil flutters at point \(A\), where the Mach number is \(M = 0.3\) and the mass ratio is \(\mu = 25\). Now consider the compatibility relationship between the reduced velocity, \(V\), and Mach number, \(M\):

\[
V = U \frac{1}{\omega_0 b} = M \sqrt{\frac{T}{T_0}} \frac{a_0}{\omega_0 b}.
\]

Eq. (2) defines the ratio \(a_0/\omega_0 b\) at point \(A\) which is denoted by the \(a_0/3\) line in Fig. 1a for the given Mach
Fig. 2  Comparison of Computational Results with those from Wind Tunnel Testing. a) Flutter Speed Surface vs Mach Number and Mass Ratio with the Numerically Simulated Wind Tunnel Trajectory as Indicated, b) Inverse Flutter Mass Ratio vs Mach Number, c) Flutter Reduced Frequency and Frequency vs Mach Number, d) Flutter Mode vs Mach Number. (NACA 0012, $\omega_\| / \omega_\alpha = 0.646$, $x_\alpha = 0$, $r_{20}^* = 1.024$, $a = 0$.)

A gain of the altitude during a flight will cause the mass ratio to grow from its value at $A$ to that at point $P$. Now to reach the flutter condition at this altitude, the pilot would have to increase the Mach number from $M = 0.3$ (p. $P$) to $M = 0.65$ (p. $B$).

In the literature the flutter boundary is often represented by the flutter speed index (the ratio of the nondimensional flutter speed and the square root of mass ratio) which here will be denoted by $V^*$. In the following, for the three-dimensional parameter flutter surfaces the Mach number axis is often pointed in the left direction in order to provide a better view of the “transonic dip” region.

A flutter speed index surface corresponding to Fig. 1a is shown in Fig. 1b. In addition to the flight test trajectory $AB$, Fig. 1b shows the flutter trajectory $AC$ originating from point $A$. $AC$ corresponds to a constant mass ratio path on the flutter surface. These two trajectories, when projected on $V^*$ versus $M$ plane as in Fig. 1c, appear dramatically different from each other. For the “flight test” curve, the intersection of the flutter and $\Sigma$ surfaces beyond point $B$ occurs with little increase in Mach number (and therefore flutter velocity) while the mass ratio keeps growing steadily. Near $M = 0.8$ the trajectory becomes nearly parallel with the $\mu$ axis thus causing $V^*$ in Fig. 1c to go rapidly to zero. (At $M = 0.8$ the calculated mass ratio is $\mu \approx 45000$.) The flutter frequency at that point is essentially the same as the coupled in vacuo natural frequency corresponding to a dominant
plunge motion, i.e. \( \omega_1/\omega_2 = 0.76 \), see Fig. 1d. Correspondingly the flutter eigenvector (not shown here) is dominated by the plunge motion. This is an example of single DOF flutter, but note the critical aeroelastic mode is a mass coupled natural mode, albeit one that is plunge dominated.

An example where AB is a possible wind tunnel flutter trajectory is considered next.

**Comparison of Wind Tunnel and Computational Results**

Ref. [6] provides detailed data from a NACA 0012 Benchmark Model wind tunnel experiment performed in the NASA Langley Transonic Dynamics Tunnel. In wind tunnel flutter testing the procedure usually consists of fixing the Mach number and then decreasing or increasing the density of the flow until the system reaches flutter. The flutter trajectory for the values of the speed of sound from Ref. [6] is marked on the computed flutter surface, see Fig. 2a. The trajectory has a gap in the Mach number range 0.82 < \( M < 0.92 \), where the compatibility surface (not shown here, described in the previous section) passes above the transonic dip “valley”.

Since the flutter velocity per se (or Mach number) is taken to be identical for both the computational and experimental models, it is advantageous for comparison of the computational and experimental results to consider \( 1/\mu_f \) instead of the flutter speed index \( V_f/\mu \). A plot of \( 1/\mu_f \) versus the Mach number is discussed next, see Fig. 2b. Note that the experimental Pitch and Plunge Apparatus (PAPA) has a rather heavy mass of 192 lbs for a 32 \times 16 \text{ in}^2 \) wing.

In the range 0.30 \( \leq M \leq 0.82 \), there is good agreement between the numerical and experimental results.

In the range 0.82 < \( M < 0.88 \), no (experimental) flutter data have been presented in Ref. [6]. As has already been mentioned, results of the Euler HB method reveal that in the range 0.82 < \( M < 0.92 \) there is no theoretical flutter boundary for the considered values of the mass ratio and speed of sound. The dotted lines (at \( M = 0.82 \) and 0.92) that go down to low values of
Fig. 4  NACA 0004 and 0012 Airfoils Flutter Similarity Results. a) Transonic Similarity Parameter, b) Reduced Frequency, c) Amplitude Ratio, and d) Phase. All plotted vs Mach Number that corresponds to NACA 0004 Airfoil. ($\psi = 0\text{.}88$, $x_o = 0\text{.}25$, $r_o = 0\text{.}75$, $a = 0\text{.}0$).

$1/\mu_f$ are the expected results for values of $\mu$ beyond those considered in the numerical study. It is also possible that at very high mass ratios the value of $V_f$ in the transonic dip “valley” region (Fig. 2a) could rise enough to intersect with the compatibility surface in that Mach number range making the dotted lines in Fig. 2b connect at very low values of $1/\mu_f$.

In the range $0.88 \leq M \leq 0.95$, Rivera and his colleagues observed a “plunge instability”, where “the flutter motion consisted of primarily the plunge mode.” The Euler HB method branch in this range of Mach number starts at higher Mach numbers and the $1/\mu_f$ values are larger than those of the experiment. The reason for these disagreements between experiment and theory is believed to be the influence of viscous effects (which are not accounted for in Euler aerodynamic models): “Flow Visualization using tufts indicated strong shock-induced separation for this Mach number range”.$^6$. Moreover, viscosity increases the effective airfoil thickness as perceived by the flow. Increased thickness causes a shift of the transonic dip to lower Mach numbers and also for a given mass ratio it lowers the flutter velocity at Mach numbers above the transonic dip. The effect of airfoil thickness per se on flutter is studied in the next section.

In the $0.30 \leq M \leq 0.82$ range, frequency and reduced frequency of flutter results correlate well between the computational and wind tunnel models, see Fig. 2c. (No experimental flutter frequency or mode data were recorded for high transonic Mach numbers in Ref. [6].) The flutter frequency results show that this is a coalescence flutter, with a dominating plunge mode. This is also seen in Fig. 2d, where the flutter eigenvector is shown.

Aerodynamic and Aeroelastic Similarity

Using the full potential equation Bendiksen$^7$ derived similarity parameters for unsteady transonic flows. These similarity parameters allow a comparison of equivalent flows over airfoils of the same families, but
with different thicknesses and operating at different Mach numbers. One basic aerodynamic similarity variable is:

\[
\chi = \frac{1 - M^2}{[(\gamma + 1)M^2\delta]^{2/3}} \tag{3}
\]

Bendiksen noted that in the limit of steady flow \( \chi \) differs slightly from the classical von Karman result, but \( \chi \) is the same similarity parameter as obtained by Spreiter by fitting experimental data.

The other similarity parameter used here in order to test aerelastic similarity rules for prediction of a flutter boundary shift with different airfoil thicknesses is the transonic flutter similarity parameter, \( \psi \),

\[
\psi = \frac{V_u^2}{\pi [(\gamma + 1)M^2]^{1/3} \delta^{1/3 - \epsilon(\omega)}} \tag{4}
\]

“The form and magnitude of the exponent \( \epsilon(\omega) \) can be established analytically, by numerical (CFD) calculations, or deduced from wind tunnel tests on aeroelastic models,” wrote Bendiksen. Here Eqs. (4,3) are applied to NACA 0004 and NACA 0012 airfoils.

**NACA 0004 and 0012 Similarity Results**

The flutter speed index surfaces for both NACA 0004 and NACA 0012 airfoils are shown in Fig. 3a and b. As expected, for the thicker airfoil the transonic dip occurs at the lower Mach numbers.

From Eq. (3) the aerodynamically similar Mach numbers for these airfoils satisfy the relationship,

\[
1 - M_{12}^2 = \left( \frac{M_{04}}{M_{12}} \right)^{4/3} = 3^{2/3}, \tag{5}
\]

where the subscript indicates the airfoil thickness to chord percent ratio. Fig. 3c shows the lift slope coefficient as a function of Mach number. This figure shows that the lift slope coefficient peaks align for aerodynamically similar Mach numbers.

The difference in the flutter speed index for \( \mu = 50 \) and 5000 is shown in Fig. 3d. From these results one can conclude that, due to the alignment of the lift slope coefficients, the transonic dips also become aligned for aerodynamically similar Mach numbers.

The next step is to test the aerelastic similarity rule, Eq. (4). This is done in Fig. 4a, where the transonic flutter similarity parameter is shown versus aerodynamically similar Mach number (for the case of \( \epsilon(\omega) = 0 \)). In the transonic range of Mach numbers, the flutter similarity parameters for these two airfoils are close until the Mach numbers reach \( M_{\delta=04} \approx 0.95 \) at the end of the transonic dip. Flutter reduced frequencies, amplitude ratios and phases are shown in Figs. 4b, c, and d respectively. With the exception of the flutter reduced frequency and eigenmode for Mach numbers above the transonic dip, the equivalent similarity quantities in Fig. 4b-d agree well.

**Structural-Inertial and Geometrical Parameter Sensitivity**

In functional form and to emphasize the parameters involved, the aeroelastic equations can be written as,

\[
\begin{bmatrix}
F(a, x_{\alpha}, r_{\alpha}, \text{geom}, M, \omega, \mu, \frac{\omega_h}{\omega_\alpha}, V)
\end{bmatrix}
\begin{bmatrix}
h/b
\end{bmatrix}
= 0.
\]

In the following several sections, the effects of these parameters on the flutter speed index are briefly studied.

**Static Unbalance Effect**

Dependence of the flutter speed index on the airfoil static unbalance, \( x_{\alpha} \equiv S_{\alpha}/mb = x_{c.g.}/b \) is shown in Fig. 5.

When the elastic axis is behind or coincides with the center of gravity, \( x_{c.g.} \), (i.e. \( x_{\alpha} \leq 0 \)) for Mach numbers up to \( M = 0.88 \), the flutter occurs at very low reduced frequencies \( 0.01 \leq \tilde{\omega} \leq 0.05 \) and the flutter speed index is very high, see Fig. 5a. As the Mach number exceeds \( M \geq 0.89 \) the reduced frequency of flutter rises to values of \( 0.35 \leq \tilde{\omega} \leq 0.55 \) and the flutter speed index drops to the values below one. For Mach numbers \( M \geq 0.95 \), the computations showed no flutter boundary for low mass ratios.

For small positive values of \( x_{\alpha} \), e.g. \( x_{\alpha} = 0.10 \), it is observed that in the vicinity of \( M \sim 0.89 \) there is a jump in the flutter speed index. As in the case of a negative or zero \( x_{\alpha} \), for \( x_{\alpha} = 0.10 \) for \( M \geq 0.95 \), the computations showed no flutter boundary for low mass ratios. For \( x_{\alpha} \geq 0.25 \) the flutter speed index remains below one in the range of Mach numbers considered here. Note the transonic dip becomes more evident as \( x_{\alpha} \) grows from 0.25 to 1.00.

**Radius of Gyration Effect**

Dependence of the flutter speed index on the radius of gyration of airfoil about the elastic axis, \( r_{\alpha}^2 \equiv I_{\alpha}/mb^2 \), is shown in Fig. 6. These results show that an increase in the radius of gyration tends to increase the flutter speed index especially for higher mass ratios. A similar conclusion was reached for a different airfoil geometries (results not shown).

**Mass Ratio and Ratio of Uncoupled Natural Frequencies Effect**

All the flutter velocity results presented in this paper show a relatively weak dependence on the mass ratio per se and as \( \mu \) is increased the flutter speed index is in general decreasing.

Dependence of the flutter speed index on the bending-torsion frequency ratio, \( \omega_h/\omega_\alpha \), is shown in Fig. 7.

These results indicate that the flutter speed index is sensitive to the ratio of uncoupled natural frequencies. For example, comparing Fig. 7a with b one can observe an overall decrease of \( V_\mu \) as the values of \( \omega_h/\omega_\alpha \) is increased from 0.5 to 0.8. Of course it is known that
Fig. 5  Static Unbalance Effect. Flutter Speed Index vs Mach Number and Mass Ratio. a) \( x_\alpha = 0.00 \), b) \( x_\alpha = 0.10 \), c) \( x_\alpha = 0.25 \) and d) \( x_\alpha = 1.00 \). (NACA 65A004, \( \omega_h/\omega_\alpha = 0.8, r^{2}_\alpha = 0.75, a = -0.6 \).)

Fig. 6  Radius of Gyration Effect. Flutter Speed Index vs Mach Number and Mass Ratio. a) \( r^{2}_\alpha = 0.25 \), and b) \( r^{2}_\alpha = 4.00 \). (NACA 65A004, \( \omega_h/\omega_\alpha = 0.8, x_\alpha = 0.25, a = -0.6 \).)
the flutter speed index has a minimum near $\omega_h/\omega_n = 1$. For a detailed discussion of the dependence of flutter results on the mass ratio and the ratio of uncoupled natural frequencies, the reader is referred to a recent work of the authors on this subject\(^2\).

**Airfoil Geometry Effect**

Dependence of the flutter speed index on the geometry of airfoil is shown for NACA 0004 and NACA 65A004 airfoils in Fig. 8. Both airfoils have 4\% thickness-to-chord ratio but belong to two different families of NACA airfoils. One of the main differences in the geometry of these airfoils is the location of the point of maximum thickness. This point is shifted aft for the NACA 65A004 airfoil compared to the NACA 0004 airfoil, see Fig. 8a. Fig. 8b shows the lift curve slope coefficients for these two airfoils. Note that the peaks in Fig. 8b occur in the range $0.92 < M < 0.94$. The relative difference in flutter speed index of these airfoils is shown in Fig. 8c. Note that this difference is the most dramatic (up to is 45\% at some mass ratios) at Mach number $M = 0.95$, i.e. at a Mach number higher than those where the peaks in the lift curve slope coefficients occur. This can be seen better in Fig. 8d.

**Location of Elastic Axis Effect**

Dependence of the flutter speed index on the location of elastic axis, $a = e/b$, is shown in Fig. 9. The elastic axis for NACA 0004 airfoil was placed at one-fifth, $a = -0.6$, and also the middle, $a = 0.0$, of the chord. The flutter speed index for the case of mid-chord elastic axis location is shown in Fig. 9a. (The respective result for $a = -0.6$ was shown before in Fig. 3a.) The relative difference in flutter speed index for the two cases is shown in Fig. 9b. The results indicate that moving the elastic axis aft from $a = -0.6$ to $a = 0.0$ decreases the flutter speed index in subsonic Mach numbers by 30\%. The largest difference (approximately 50\%) in the flutter speed index occurs at Mach numbers close to $M = 1$.

**Conclusions**

In this paper, the advantages of presenting the transonic flutter velocity as a function of the Mach number and mass ratio in a three-parameter-map are demonstrated. This is particularly useful for simulating a wind tunnel flutter trajectory or doing a flutter parameter study. For example, in the case of comparing computational results with experimental data such a map provides a much clearer understanding of where an experimental system flutters at a given speed of sound and Mach number. Numerical results generally agreed well with the experiment.

Flutter similarity for airfoils of different thickness was considered to test recent developments on this subject\(^7\). Equivalent transonic flutter parameters for NACA 0004 and NACA 0012 airfoils agreed well except for (equivalent) Mach numbers above the transonic dip.

**References**

Fig. 8  Airfoil Geometry Effect for NACA 0004 and NACA 65A004 Airfoils. a) Airfoils Geometry, b) Ratio of Steady Lift to Pitch Angle vs Mach Number, c) Relative Difference in Flutter Speed Index, $\Delta V_{\mu \text{ geometry}} \equiv (V_{\mu \text{NACA0004}} - V_{\mu \text{NACA65A004}})/V_{\mu \text{NACA0004}}$, vs Mach Number and Mass Ratio, d) Flutter Speed Index vs Mach Number at $\mu = 100$. ($\alpha = 0.25$, $r_0^2 = 0.75$, $a = -0.6$, $\omega_h/\omega_0 = 0.8$.)

Fig. 9  Location of Elastic Axis Effect. a) Flutter Speed Index vs Mach Number and Mass Ratio for $a = 0.0$; b) Relative Difference in Flutter Speed Index, $\Delta V_{\mu \text{ e.a.}} \equiv (V_{\mu a=-0.6} - V_{\mu a=0})/V_{\mu a=-0.6}$, vs Mach Number and Mass Ratio. (NACA 0004, $x_\alpha = 0.25$, $r_0^2 = 0.75$, $\omega_h/\omega_0 = 0.8$.)

9 of 10

